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Supersymmetric Pair of *q*-Deformed Nonlocal Operators

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A simple version of the q-deformed calculus is used to generate a pair of q-nonlocal, second-order difference operators by means of deformed counterparts of Darboux intertwining operators for the Schrödinger–Hermite oscillators at zero factorization energy. These deformed nonlocal operators may be considered as supersymmetric partners and their structure contains contributions originating in both the Hermite operator and the quantum harmonic oscillator operator. There are also extra $\pm x$ contributions. The undeformed limit, in which all q-nonlocalities wash out, corresponds to the usual supersymmetric pair of quantum mechanical harmonic oscillator Hamiltonians. The more general case of negative factorization energy is briefly discussed as well.

In this work, I present a simple q-deformed procedure [1] for the basic case of the one-dimensional quantum harmonic oscillator, by which I build a 'supersymmetric' pair of q-nonlocal operators possessing terms whose $q \to 1$ limits belong to either the Hermite polynomial operator or the Schrödinger quantum oscillator operator. There are $\pm x$ extra terms as well. The procedure is based on the idea of using as fundamental tools a sort of deformed counterpart of the intertwining operators encountered in the area of Darboux transformations (see ref. 2 for review). I shall use their factorization property to get the q-deformed second-order operators which, being q-nonlocal, may be considered as more general than both the usual Hermite one and the quantum mechanical harmonic oscillator operator.

The standard Hermite operator $\hat{O}_H$ reads ($D = d/dx$)

$$\hat{O}_H = D^2 - 2xD + 2n \tag{1}$$

and gives rise to the equation for the Hermite polynomials $H_n(x)$, $\hat{O}_H H_n(x)$

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$= 0$. Writing $\hat{O}_H = D^2 - 2xD - 2 + 2(n + 1)$, I shall treat $D^2 - 2xD - 2$ as the Fokker–Planck (FP) part of the Hermite operator for a stationary transition-probability density since $-2xD$ corresponds to the $(dU/dx)D$ drift contribution in the FP stationary operator (drift potential $U = -x^2$), whereas -2 stands for the d^2U/dx^2 contribution of the FP drift. The last term $2(n + 1)$ gives the departure of the Hermite operator from the corresponding FP stationary operator for which polynomial oscillations are not allowed, and in fact is responsible for turning the FP interpretation into a formal one and not a physical one. As is well known for this basic case, by means of the functions $\phi_n = [\exp(-x^2/2)]H_n(x)$ one can go to the operator

$$\hat{O}_\phi = -D^2 + [x^2 - (2n + 1)] \tag{2}$$

which, in the ϕ_n space, is essentially the Schrödinger quantum harmonic oscillator operator up to a scaling, choice-of-units factor. One should notice that this usage of the ϕ_n functions leads to the loss of one half of the d^2U/dx^2 drift contribution. The remaining half gets the famous zero-point energy interpretation when the scaling $\frac{1}{2}\hat{O}_\phi$ is performed. In the FP interpretation, the latter scaling corresponds to setting the diffusion constant equal to 1/2 and provides the usual quantum mechanical harmonic oscillator wavefunctions $N_n\phi_n$, where $N_n = (2^n n!\sqrt{\pi})^{-1/2}$ is the normalization factor.

I now briefly recall that in the case of the one-dimensional Schrödinger operator within the context of supersymmetric quantum mechanics (SUSYQM) (see ref. 3 for review) the standard Darboux transformation operator reads

$$T = -t_u(x) + D = -u'(x)/u(x) + D \tag{3}$$

where the prime denotes the derivative with respect to x. When acting on the solutions $\psi_n(x)$ of the initial Schrödinger equation $h_0\psi_n(x) = E_n\psi_n(x)$, it transforms them into the solutions of another Schrödinger equation $h_1\varphi_n(x) = E_n\varphi_n(x)$, $\varphi_n(x) = N_nT\psi_n(x)$, with the same eigenvalues E_n. Henceforth, I will put the ground-state energy equal to zero, $E_0 = 0$, since this does not affect in any way the results. The new exactly solvable Hamiltonian has the form $h_1 = h_0 + \Delta V(x)$, where the potential difference is of Darboux type $\Delta V(x) = -2(\ln u)''$. The function $u = u(x)$ is a so-called transformation function, being a solution of the initial Schrödinger equation $h_0u(x) = \epsilon u(x)$, with $\epsilon \leq 0$ usually known as the factorization energy. It is well established that when $\epsilon < 0$ one can work with a nodeless transformation function by performing an analytic continuation [4]. Thus, $u(x) \neq 0$ for any value of the variable and $1/u(x)$ is not a square-integrable function. In this case $u \notin \mathcal{H}_1$ and the set $\{|\varphi_n\rangle\}$ is a complete basis in the Hilbert space $\mathcal{H}_1$ provided the initial system $\{|\psi_n\rangle\}$ is complete. The operator $T^+ = -t_u(x) - D$ provides the backward transformation $|\psi_n\rangle = N_n T^+ |\varphi_n\rangle$, and together with T allows for the following factorizations:

$$T^+T = h_0 - \epsilon, \qquad TT^+ = h_1 - \epsilon \tag{4}$$

The operators T and T^+ are well defined $\forall \psi \in \mathcal{H}_1$ and are conjugated to each other with respect to the inner product in the $\mathcal{H}_1$ space.

My purpose now is to get q-deformed second-order operators by means of deformed counterparts of the aforementioned intertwining operators. I still have to present some definitions and rules of the deformed calculus. Since the independent variable is maintained commutative, the employed version of the deformed calculus is similar to that previously used by some authors to deform the Coulomb problem [5]. Symmetric definitions of the q-number $[x]_q = (q^x - q^{-x}/q - q^{-1})$ and q-derivative

$$D_q f(x) = \frac{f(qx) - f(q^{-1}x)}{x(q - q^{-1})} \tag{5}$$

are used together with some basic rules of Jackson's calculus [1] such as $D_q x^n = [n]_q x^{n-1}$, $D_q^2 x^n = [n]_q [n-1]_q x^{n-2}$, $D_q(FG) = (D_q F)G(qx) + F(q^{-1}x)(D_q G)$ for any two functions F and G, respectively. The definition of the q-exponential is

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!} \tag{6}$$

which reduces to the usual exponential function as $q \to 1$, and moreover is invariant under $q \to q^{-1}$.

The main idea of this work is based on the following scheme. First, we employ as Darboux transformation functions deformed counterparts of the oscillator vacua $\psi_q \propto e_q(\beta x^2)$, where $\beta = \pm 1/2$ for the irregular and regular vacuum, respectively. Second, to exploit the factorization property of first-order deformed operators of the form

$$T_+^q = D_q - \frac{D_q \psi_q}{\psi_q} = D_q - \beta_q(x^2)x \tag{7}$$

$$T_-^q = -D_q - \frac{D_q \psi_q}{\psi_q} = -D_q - \beta_q(x^2)x \tag{8}$$

where

$$\beta_q(x^2) = \beta\left(\frac{q e_q(q\beta x^2) + q^{-1} e_q(q^{-1}\beta x^2)}{e_q(\beta x^2)}\right) \tag{9}$$

The form of $\beta_q(x^2)$ is a result of Jackson's calculus rules. As one can see, the T_+^q and T_-^q operators have been written by analogy to the continuous intertwining operators. A straightforward calculation gives the second-order deformed operators that can be obtained from the products $T_-^q T_+^q$ and $T_+^q T_-^q$, respectively. One gets

$$\hat{O}_b^q \equiv T_-^q T_+^q = -D_q^2 - [(\Delta\beta_q) x D_q] + [\beta_q^2(x^2) x^2]_{\to x} + [q(D_q \beta_q(x^2))x]_{\to qx} + [\beta_q(q^{-2}x^2)]_{\to qx} \qquad (10)$$

and

$$\hat{O}_f^q \equiv T_+^q T_-^q = -D_q^2 + [(\Delta\beta_q) x D_q] + [\beta_q^2(x^2) x^2]_{\to x} - [q(D_q \beta_q(x^2))x]_{\to qx} - [\beta_q(q^{-2}x^2)]_{\to qx} \qquad (11)$$

where

$$\Delta\beta_q = \beta_q(x^2) - q^{-1}\beta_q(q^{-2}x^2) \qquad (12)$$

The operators $\hat{O}_b^q$ and $\hat{O}_f^q$ may be considered as supersymmetric partners since they have been built according to the well-known SUSYQM method. At this point one should notice the interesting mixed structure of the two q-nonlocal operators that entail parts of both $\hat{O}_H$ and $\hat{O}_\phi$. The directional supscripts indicate the argument of the solution on which the nonoperatorial parts act. The two operators are nonlocal operators whose spaces of solutions are the functions $\phi_n^{(q)}(x) \propto e_q(-x^2/2) H_n^{(q)}(x)$, where the deformed Hermite polynomials can be defined by a q-deformed Rodrigues representation

$$H_n^{(q)} = (-1)^n e_q(x^2) D_q^n (e_q(-x^2)) \qquad (13)$$

Notice that the D_q terms in (10) and (11) are identical, but opposite in sign and correspond to the first derivative drift term in the Hermite differential operator. Of course, if one prefers the FP interpretation the two operators should be multiplied by (-1). Writing the finite difference $x(q - q^{-1}) = x\Delta q = \Delta_q x$, which for $q \to 1$ is assumed to be a q-scaling way of going to the infinitesimal limit dx, we see that the q drift parts go to zero, whereas in the same limit the potential and zero-point sectors take forms identical to those of the undeformed case. More precisely, the undeformed limits read

$$\hat{O}_b^1 \equiv h_0 = -D^2 + \beta_1^2 x^2 + \beta_1 \qquad (14)$$

and

$$\hat{O}_f^1 \equiv h_1 = -D^2 + \beta_1^2 x^2 - \beta_1 \qquad (15)$$

(14) and (15) are the usual quantum mechanical supersymmetric partner Hamiltonians for this case.

In SUSYQM terminology, only the case of zero factorization energy $\epsilon = 0$ has been tackled up to now, but following a suggestion of Bagrov and Samsonov [4], there is no difficulty to sketch the procedure for the more general case $\epsilon < 0$. First, the deformed Schrödinger solution corresponding to the excited harmonic oscillator states can be sought in the form

$$\psi_n^{(q)}(x) \propto H_n^{(q)}(x) e_q(-x^2/2) \tag{16}$$

Next, in order to avoid any singularities, it is convenient to perform an *i*-rotation $x \to ix$, leading to

$$u_p^{(q)}(x) \propto H_p^{(q)}(ix) e_q(x^2/2), \qquad p = 0, 1, 2, 3, \ldots \tag{17}$$

The undeformed functions $u_p^{(1)}$ are solutions of $h_0 u_p^{(1)} = -(p+1) u_p^{(1)}$ and are nodeless on the full line for even $p = 2k$. Therefore, they have been used by Bagrov and Samsonov as Darboux transformation functions to generate a family of regular potentials, which, according to an interpretation due to Veselov and Shabat [6], has a spectrum made up of $2k + 1$ segments with equidistant levels. This immediately suggests using $u_p^{(q)}(x)$ for even $p = 2k$ as Darboux transformation functions in the deformed case. Thus, the intertwining operators can be calculated according to

$$T^q_{2k,+} = D_q - \frac{D_q u_{2k}^{(q)}}{u_{2k}^{(q)}} \qquad \text{and} \qquad T^q_{2k,-} = -D_q - \frac{D_q u_{2k}^{(q)}}{u_{2k}^{(q)}}$$

and again by exploiting the factorization property one is led to second-order deformed, nonlocal operators of more complicated formulas than (10) and (11), which are not written down here.

In conclusion, a pair of *q*-nonlocal second-order *q*-differential (*q*-difference) operators have been introduced in this work by means of a particular *q*-deformed intertwining based on *q*-deformed oscillator vacua as Darboux transformation functions. These operators display a mixed structure between the Hermite operator, to which they are similar as regards the first derivative term, and the quantum mechanical oscillator operator, to which they are similar as regards the x^2 potential and zero-energy contributions. On the other hand, they present a supplementary *q*-nonlocal $\pm x$ potential contribution with no counterpart in either the Hermite polynomial operator or the Schrödinger x^2 oscillator operator. All these features suggest many possible applications, e.g., in mesoscopic physics. A more general case corresponding to negative factorization energies of the type $\epsilon_m = -(m+1)$, where m is an even, positive integer, has also been briefly described.

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